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Perturbative versus nonperturbative dynamics of the fuzzy $S^2 \times S^2$

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Abstract: We study a matrix model with a cubic term, which incorporates both the fuzzy $S^2 \times S^2$ and the fuzzy S^2 as classical solutions. Both of the solutions decay into the vacuum of the pure Yang-Mills model (even in the large-N limit) when the coefficient of the cubic term is smaller than a critical value, but the large-N behavior of the critical point is different for the two solutions. The results above the critical point are nicely reproduced by the all order calculations in perturbation theory. By comparing the free energy, we find that the true vacuum is given either by the fuzzy S^2 or by the "pure Yang-Mills vacuum" depending on the coupling constant. In Monte Carlo simulation we do observe a decay of the fuzzy $S^2 \times S^2$ into the fuzzy S^2 at moderate N, but the decay probability seems to be suppressed at large N. The above results, together with our previous results for the fuzzy CP^2, reveal certain universality in the large-N dynamics of four-dimensional fuzzy manifolds realized in a matrix model with a cubic term.

Keywords: Nonperturbative Effects, Non-Commutative Geometry, Matrix Models.

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1. Introduction and summary

Fuzzy spheres [1], or fuzzy manifolds in general, have recently attracted much attention in various branches of particle physics. One of the motivations comes from the general expectation that the noncommutative geometry, which is characteristic to those manifolds, provides crucial links to string theory and quantum gravity. Indeed Yang-Mills theories on noncommutative geometry appear in a certain low energy limit of string theory [2]. There is also an independent observation that the space-time uncertainty relation, which is naturally realized by noncommutative geometry, can be derived from some general assumptions on the underlying theory of quantum gravity [3]. Another motivation is to use fuzzy manifolds as a regularization scheme alternative to the lattice regularization [4]. Unlike the lattice, fuzzy manifolds typically preserve the continuous symmetries of the space-time considered, and hence it is expected that the situation concerning chiral symmetry [5 – 19] and supersymmetry might be improved.

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As expected from the connection to string theory [20], fuzzy manifolds appear as classical solutions in matrix models with a Chern-Simons-like term [21 – 25] and their dynamical properties have been studied in refs. [26 – 39]. One can actually use matrix models to define a regularized field theory on the fuzzy spheres as well as on a noncommutative torus [40], which enables nonperturbative studies of such theories from first principles [41]. These matrix models belong to the class of the so-called dimensionally reduced models (or large-N reduced models), which is widely believed to provide a constructive definition of superstring and M theories [42, 43]. In fact there are certain evidences in the IIB matrix model [43] that *four-dimensional* space-time is generated dynamically [44 – 47]. In refs. [44 – 46] the free energy of space-time with various dimensionality has been calculated using the gaussian expansion method, and the free energy turned out to take the minimum value for the four-dimensional space-time. In ref. [47] it was found that the fuzzy $\mathrm{S}^2 \times \mathrm{S}^2$ (but not the fuzzy S^2) is a solution to the 2-loop effective action. See also refs. [48 – 60] for related works on this issue. The fuzzy sphere is also useful [61] in the Coset Space Dimensional Reduction [62, 63].

In view of the variety of contexts in which fuzzy manifolds have been studied, we consider it important to study their nonperturbative dynamics from first principles by Monte Carlo simulations. In ref. [64] we have studied the dimensionally reduced 3d Yang-Mills-Chern-Simons model, which incorporates the fuzzy S^2 as a classical solution [22]. We have observed a first-order phase transition as we vary the coefficient (α) of the Chern-Simons term. For small α the large-N behavior of the model is the same as in the pure Yang-Mills model, whereas for large α a single fuzzy S^2 appears dynamically. The emergence of a fuzzy sphere in matrix models may be regarded as a prototype of the dynamical generation of space-time since it has lower dimensionality than the original dimensionality that the model can actually describe. In ref. [65] we have performed the all order calculations in perturbation theory around the fuzzy S^2 solution following the proposal in ref. [34], and confirmed that the Monte Carlo results for various observables in the "fuzzy sphere phase" can be nicely reproduced. For obvious reasons it is interesting to extend these works to *four*-dimensional fuzzy manifolds. In ref. [66] we have studied a matrix model incorporating the fuzzy S^4, and in ref. [67] a matrix model, which incorporates the fuzzy CP^2 as well as the fuzzy S^2.

In this paper we study the fuzzy $\mathrm{S}^2 \times \mathrm{S}^2$, which has been studied extensively in the literature [47, 32, 34, 36 – 39]. We perform perturbative and nonperturbative studies of a 6d Yang-Mills model with a cubic term, which incorporates the fuzzy $\mathrm{S}^2 \times \mathrm{S}^2$ as well as the fuzzy S^2 as classical solutions. Both of the solutions become unstable (even in the large-N limit) when the coefficient of the cubic term is smaller than a critical value, but the large-N behavior of the critical point is different for the two solutions. The results above the critical point are nicely reproduced by the all order calculations in perturbation theory. By comparing the free energy, we find that the true vacuum is given either by the fuzzy S^2 or by the pure Yang-Mills vacuum depending on the coupling constant. In Monte Carlo simulation we do observe a decay of the fuzzy $\mathrm{S}^2 \times \mathrm{S}^2$ into the fuzzy S^2 at moderate N, but the decay probability seems to be suppressed at large N.

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In fact the above results are qualitatively similar to the fuzzy CP^2 case as far as large-N properties are concerned. This reveals certain universality in four-dimensional fuzzy manifolds realized in a matrix model with a *cubic* term. In the case of the fuzzy S^4 [66] we had to add a *quintic* term, which led to a totally different situation. We note, however, that even within matrix models with a cubic term, we have found two directions which lead to qualitatively different results. In ref. [68] it is shown that an additional "mass term" can make various fuzzy sphere solutions almost degenerate at the classical level, and it is possible that the true quantum vacuum is described by a set of coincident fuzzy spheres, giving rise to nontrivial gauge groups. In ref. [69] we have shown that supersymmetry removes the quantum effects, and as a result the single fuzzy sphere becomes always stable if the large-N limit is taken in such a way that various correlation functions scale.

This paper is organized as follows. In section 2 we define the model and discuss its classical solutions. In sections 3 and 4 we study the properties of the fuzzy $S^2 \times S^2$ and the fuzzy S^2, respectively, by performing Monte Carlo simulations and the all order calculations in perturbation theory. In section 5 we determine the true vacuum of the model based on the comparison of the free energy. In section 6 we discuss a transition from the fuzzy $S^2 \times S^2$ to the fuzzy S^2 observed in Monte Carlo simulation. In the appendices we provide the details of our calculations.

Note added. Part of this work has been reported by one of the authors at YITP workshop "Quantum Field Theory 2004", July, 2004, at a meeting of the Physical Society of Japan, Kochi, Sep. 2004 and at Dublin Institute for Advanced Studies, Dec. 2004. While we were preparing this article, we received a preprint [39], which discusses the phase diagram of a similar model based on the one-loop effective action.

2. The model and its classical solutions

The model we study in this paper is defined by the action

$$S[A] = N \operatorname{tr}\left(-\frac{1}{4}[A_\mu, A_\nu]^2 + \frac{2}{3}\, i\,\alpha\, f_{\mu\nu\rho}\, A_\mu A_\nu A_\rho\right), \tag{2.1}$$

where A_μ ($\mu = 1, \cdots, 6$) are traceless $N \times N$ hermitean matrices. Here and henceforth we assume that repeated Greek indices are summed over all possible integers. The rank-3 tensor $f_{\mu\nu\rho}$ in the cubic term is given by

$$f_{\mu\nu\rho} = \begin{cases} \epsilon_{\mu\nu\rho} & \text{for } \mu,\nu,\rho = 1,2,3\,, \\ \epsilon_{\mu\nu\rho} & \text{for } \mu,\nu,\rho = 4,5,6\,, \\ 0 & \text{otherwise}\,, \end{cases} \tag{2.2}$$

where $\epsilon_{\mu\nu\rho}$ is a totally antisymmetric tensor with $\epsilon_{123} = \epsilon_{456} = 1$.

The classical equation of motion reads

$$[A_\nu, [A_\mu, A_\nu]] - i\,\alpha\, f_{\mu\nu\rho}\,[A_\nu, A_\rho] = 0\,. \tag{2.3}$$

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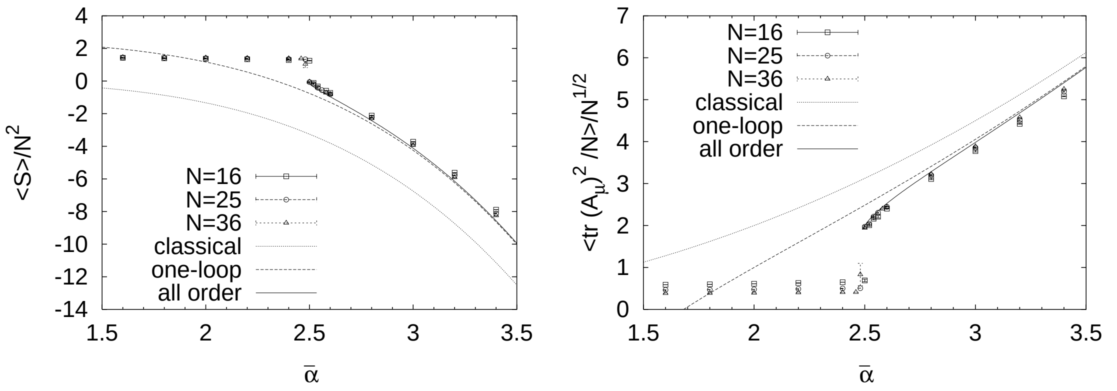

Figure 1: The observables obtained by Monte Carlo simulation with the fuzzy $S^2 \times S^2$ start are plotted against $\bar{\alpha} = \alpha N^{\frac{1}{4}}$ for $N = 16, 25, 36$ (i.e., $n = 4, 5, 6$). The dotted (dashed) lines represent the classical (one-loop) results at large N. The solid lines represent the all order results at large N obtained above the critical point.

In addition to the commutative solution, which exists also for $\alpha = 0$, it has the fuzzy $S^2 \times S^2$ solution, which is defined by

$$A_\mu = X_\mu^{(n_1,n_2;k)} \stackrel{\text{def}}{=} \begin{cases} \alpha \left(L_\mu^{(n_1)} \otimes \mathbf{1}_{n_2} \right) \otimes \mathbf{1}_k & \text{for } \mu = 1,2,3\,, \\ \alpha \left(\mathbf{1}_{n_1} \otimes L_\mu^{(n_2)} \right) \otimes \mathbf{1}_k & \text{for } \mu = 4,5,6\,, \end{cases} \tag{2.4}$$

where $L_\mu^{(n)}$ represents the n-dimensional representation of the SU(2) Lie algebra, and the integers n_1, n_2 and k should satisfy $N = n_1\, n_2\, k$. Let us define the "Casimir operators"

$$Q_1 = \sum_{\mu=1}^{3} (A_\mu)^2\,, \qquad Q_2 = \sum_{\mu=4}^{6} (A_\mu)^2\,. \tag{2.5}$$

Plugging the solution (2.4), we obtain $Q_j = R_j \mathbf{1}_N$, where $R_j = \frac{1}{2}\alpha\sqrt{(n_j)^2 - 1}$ for $j = 1, 2$. This classical solution (2.4) therefore describes k coincident fuzzy $S^2 \times S^2$ with the radii R_1 and R_2 in the 123- and 456- directions, respectively. If we expand the model around the solution, we obtain the U(k) gauge theory on the fuzzy $S^2 \times S^2$ generalizing the work on the fuzzy S^2 [22]. The classical action for the solution is given by

$$S\left[X^{(n_1,n_2;k)}\right] = -\frac{N^2\alpha^4}{24}\left\{(n_1)^2 + (n_2)^2 - 2\right\}. \tag{2.6}$$

In what follows we focus on the symmetric fuzzy $S^2 \times S^2$ ($n_1 = n_2$) and the fuzzy S^2 ($n_2 = 1$). Monte Carlo simulation is performed using the heat bath algorithm as in refs. [49, 64].

3. Properties of the fuzzy $S^2 \times S^2$

In this section we study the properties of the single ($k = 1$) fuzzy $S^2 \times S^2$. We perform Monte Carlo simulation taking $A_\mu = X_\mu^{(n,n;1)}$ as the initial configuration, where $n = \sqrt{N}$. Let us consider the expectation value of the action (2.1) and the "extent of space-time"

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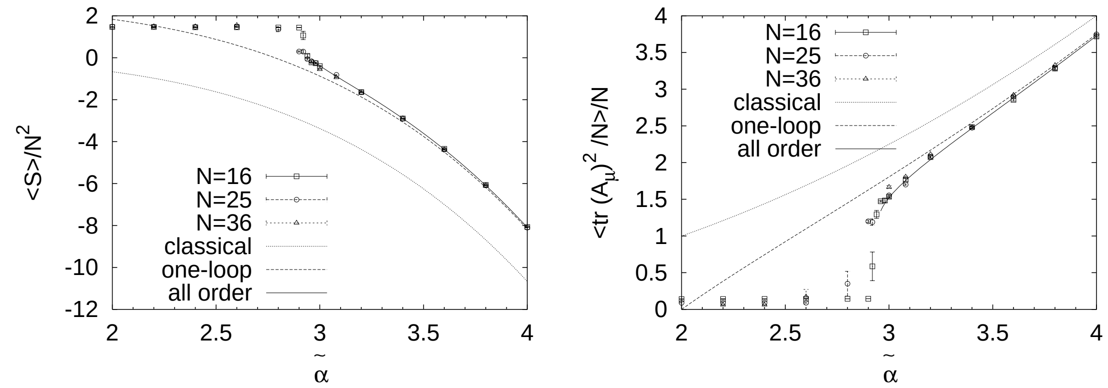

Figure 2: The observables obtained by Monte Carlo simulation with the fuzzy S^2 start are plotted against $\tilde{\alpha} = \alpha N^{\frac{1}{2}}$ for $N = 16, 25, 36$. The dotted (dashed) lines represent the classical (one-loop) results at large N. The solid lines represent the all order results at large N obtained above the critical point.

$\frac{1}{N}\,\mathrm{tr}(A_\mu)^2 = \frac{1}{N}\,\mathrm{tr}\,(Q_1{+}Q_2)$. In figure 1 we plot these quantities obtained by the simulation with $N = 16, 25, 36$ against $\bar{\alpha} = \alpha\, N^{\frac{1}{4}}$. We observe a discontinuity at $\bar{\alpha} \simeq 2.5$, which implies the existence of a first-order phase transition. The critical point agrees with the result (B.25) obtained analytically from the effective action.

Above the critical point we can calculate the observables in the large-N limit to all orders in perturbation theory as

$$\frac{1}{N^2}\langle S\rangle \simeq -\frac{\bar{\alpha}^4}{12} + \frac{5}{2} + \frac{8}{\bar{\alpha}^4} + \frac{448}{3\bar{\alpha}^8} + \frac{3520}{\bar{\alpha}^{12}} + \cdots\,, \tag{3.1}$$

$$\frac{1}{\sqrt{N}}\left\langle \frac{1}{N}\,\mathrm{tr}(A_\mu)^2 \right\rangle \simeq \frac{\bar{\alpha}^2}{2} - \frac{4}{\bar{\alpha}^2} - \frac{40}{\bar{\alpha}^6} - \frac{768}{\bar{\alpha}^{10}} - \frac{18304}{\bar{\alpha}^{14}} - \cdots\,. \tag{3.2}$$

(See appendix B.4 for derivation.) This result, as well as the classical and one-loop results, is plotted in figure 1. We observe that the Monte Carlo data for $\bar{\alpha} > \bar{\alpha}_{\mathrm{cr}}$ approach the all order results as N increases.

4. Properties of the fuzzy S^2

Next we study the properties of the single ($k = 1$) fuzzy S^2. We perform Monte Carlo simulation taking $A_\mu = X_\mu^{(N,1;1)}$ as the initial configuration. In figure 2 we plot the results for $N = 16, 25, 36$ against $\tilde{\alpha} = \alpha N^{\frac{1}{2}}$. We observe a gap at the critical point $\tilde{\alpha} \simeq 3.0$, which agrees with the result (C.8) obtained from the effective action. The Monte Carlo results above this point agree well with the all order results in perturbation theory at large N, which are given by

$$\frac{1}{N^2}\langle S\rangle \simeq -\frac{\tilde{\alpha}^4}{24} + \frac{5}{2} + \frac{16}{\tilde{\alpha}^4} + \frac{1792}{3\tilde{\alpha}^8} + \frac{28160}{\tilde{\alpha}^{12}} + \cdots\,, \tag{4.1}$$

$$\frac{1}{N}\left\langle \frac{1}{N}\,\mathrm{tr}(A_\mu)^2 \right\rangle \simeq \frac{\tilde{\alpha}^2}{4} - \frac{4}{\tilde{\alpha}^2} - \frac{80}{\tilde{\alpha}^6} - \frac{3072}{\tilde{\alpha}^{10}} - \frac{146432}{\bar{\alpha}^{14}} - \cdots\,. \tag{4.2}$$

(See appendix C.3 for derivation.)

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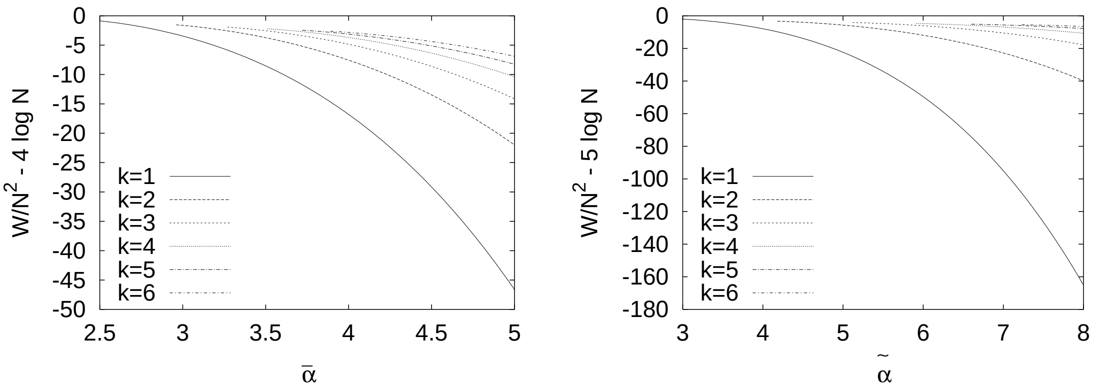

Figure 3: The free energy for the k coincident fuzzy $\mathrm{S}^2 \times \mathrm{S}^2$ (left) and the k coincident fuzzy S^2 (right) is plotted for $k = 1, 2, \cdots, 6$. We have subtracted the irrelevant constants ($4 \log N$ and $5 \log N$, respectively) to make the quantity finite in the large-N limit.

5. The true vacuum

In this section we determine the "true vacuum" by comparing the free energy, which can be obtained to all orders in perturbation theory as

$$\frac{1}{N^2} W_{k\,\mathrm{S}^2\times\mathrm{S}^2} \simeq -\frac{\bar{\alpha}^4}{12k} + 4\log\bar{\alpha} + \log\frac{16N^4}{k^2} - \frac{15}{4} - \frac{8k}{\bar{\alpha}^4} - \frac{224k^2}{3\bar{\alpha}^8} - \cdots\,, \tag{5.1}$$

$$\frac{1}{N^2} W_{k\,\mathrm{S}^2} \simeq -\frac{\tilde{\alpha}^4}{24k^2} + 4\log\tilde{\alpha} + \log\frac{N^5}{k^4} - \frac{11}{4} - \frac{16k^2}{\tilde{\alpha}^4} - \frac{896k^4}{3\tilde{\alpha}^8} - \cdots \tag{5.2}$$

for the k coincident fuzzy $\mathrm{S}^2 \times \mathrm{S}^2$ and the k coincident fuzzy S^2, respectively. (See appendices B.4 and C.3 for derivation.) Note that these solutions are stable above the critical points given by (B.25) and (C.8).

Let us first compare the free energy among different values of k. In figure 3 we plot the free energy for the fuzzy $\mathrm{S}^2 \times \mathrm{S}^2$ (left) and S^2 (right) against $\bar{\alpha}$ and $\tilde{\alpha}$, respectively, for various k. The free energy for $k > 1$ is always larger than that for $k = 1$, which implies that the dynamically generated gauge group is U(1) for both the fuzzy $\mathrm{S}^2 \times \mathrm{S}^2$ and the fuzzy S^2 cases.

We therefore take $k = 1$ and compare the free energy for the fuzzy $\mathrm{S}^2 \times \mathrm{S}^2$ and the fuzzy S^2. In order for the single fuzzy $\mathrm{S}^2 \times \mathrm{S}^2$ to be stable, we need to have $\bar{\alpha} > \bar{\alpha}_{\mathrm{cr}}^{(k=1\,\mathrm{S}^2\times\mathrm{S}^2)}$. In that regime, however, $\tilde{\alpha} \simeq \mathrm{O}(N^{1/4})$ and therefore $W_{k=1\,\mathrm{S}^2} \simeq -\mathrm{O}(N^3)$, meaning that the single fuzzy S^2 has much smaller free energy than the single fuzzy $\mathrm{S}^2 \times \mathrm{S}^2$.

As we have done in ref. [68], we can also calculate the free energy W_{YM} in the Yang-Mills phase for $\alpha \to 0$ in the large-N limit since it should agree with the free energy for the pure Yang-Mills model ($\alpha = 0$) studied in ref. [71]. We obtain (See appendix D.)

$$\frac{1}{N^2} W_{\mathrm{YM}} = 3\log N + 6\left\{-0.33 + \log\left(3^{1/4}\pi^{-1/2}\right)\right\}\,. \tag{5.3}$$

Comparing this result with (5.2) for $k = 1$, we find that the true vacuum of this model is given by the single fuzzy S^2 for $\tilde{\alpha} > \tilde{\alpha}_{\mathrm{cr}}$ and by the pure Yang-Mills vacuum for $\tilde{\alpha} < \tilde{\alpha}_{\mathrm{cr}}$,

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where the critical point is

$$\tilde{\alpha}_{\rm cr} \simeq (48 \log N)^{1/4} . \tag{5.4}$$

This result is *exact* at large N since the calculation of $W_{k=1\,\mathrm{S}^2}$ and $W_{\rm YM}$ are both reliable at the critical point $\tilde{\alpha}_{\rm cr}$.

Note that the *lower* critical point $\tilde{\alpha}_{\rm cr}^{(\rm l)} \equiv \tilde{\alpha}_{\rm cr}^{(k=1\,\mathrm{S}^2)}$, at which the single fuzzy S^2 becomes unstable, is below $\tilde{\alpha}_{\rm cr}$. On the other hand, from Monte Carlo simulations we find that the *upper* critical point, at which the pure Yang-Mills vacuum becomes unstable, is $\alpha_{\rm cr}^{(\rm u)} \simeq 1.51$, which is above $\alpha_{\rm cr} \equiv \frac{1}{\sqrt{N}}\tilde{\alpha}_{\rm cr}$. The existence of such three kinds of critical points is typical to first-order phase transitions. In the region $\alpha_{\rm cr}^{(\rm l)} < \alpha < \alpha_{\rm cr}^{(\rm u)}$ we observe a hysteresis behavior in simulations similarly to the one observed in ref. [64].

6. A transition from the fuzzy $\mathrm{S}^2 \times \mathrm{S}^2$ to the fuzzy S^2

For $\alpha > \alpha_{\rm cr}$, since the true vacuum is given by the fuzzy S^2, we may be able to see the fuzzy $\mathrm{S}^2 \times \mathrm{S}^2$ prepared as the initial configuration decay into the fuzzy S^2 in Monte Carlo simulations if we make a very long run.

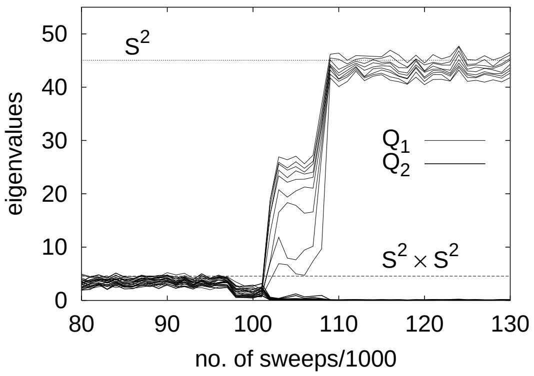

Figure 4: The eigenvalues of the "Casimir operators" Q_1 and Q_2 are plotted against the number of the sweeps in Monte Carlo simulation with the fuzzy $\mathrm{S}^2 \times \mathrm{S}^2$ start. The dashed (dotted) line represents the classical value for the fuzzy $\mathrm{S}^2 \times \mathrm{S}^2$ (the fuzzy S^2).

In order to distinguish the classical solutions, it is convenient to look at the eigenvalues of the "Casimir operators" Q_1 and Q_2. In figure 4 we plot the eigenvalues obtained for $N = 9$ and $\bar{\alpha} = 2.6$ (barely above $\bar{\alpha}_{\rm cr}^{(k=1\,\mathrm{S}^2\times\mathrm{S}^2)}$) against the number of the sweeps of the heat-bath algorithm taking $A_\mu = X_\mu^{(n,n;1)}$ as the initial configuration. For the first 97,000 sweeps all the eigenvalues of Q_1 and Q_2 are fluctuating around the classical value $(R_1)^2 = (R_2)^2 \simeq 4.5$ for the symmetric fuzzy $\mathrm{S}^2 \times \mathrm{S}^2$, but after a while the eigenvalues of Q_1 and Q_2 are split into the classical values $(R_1)^2 \simeq 45$ and $(R_2)^2 \simeq 0$ for the fuzzy S^2, which indicates that the configuration has become the single $(k = 1)$ fuzzy S^2. In this particular event we observe that the fuzzy S^2 is formed in the 123-directions, but the formation of the fuzzy S^2 in the 456-directions should be observed with equal probability due to the Z_2 symmetry for exchanging the two sets of directions.

We have performed a simulation for $N = 16$ with the same $\bar{\alpha} = 2.6$ starting again from the fuzzy $\mathrm{S}^2 \times \mathrm{S}^2$, but the decay is not observed within 10,000,000 sweeps. This suggests that the potential barrier between the fuzzy $\mathrm{S}^2 \times \mathrm{S}^2$ and the fuzzy S^2 grows with N, and therefore the decay probability is suppressed at large N.

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A. Evaluation of the free energy around a classical solution

In this section we formulate the perturbation theory and derive a formula for the free energy. Let us evaluate the partition function $Z = \int dA\, e^{-S}$ around the fuzzy $\mathrm{S}^2 \times \mathrm{S}^2$ solution $X_\mu^{(n_1,n_2;k)}$ at the one-loop level. The measure for the path integral is defined by $dA = \prod_{\mu=1}^{6} \prod_{a=1}^{N^2-1} dA_\mu^a$, where $A_\mu = \sum_{a=1}^{N^2-1} A_\mu^a\, t^a$ with t^a being the generators of $\mathrm{SU}(N)$ normalized by $\mathrm{tr}(t^a\, t^b) = \delta_{ab}$.

We need to fix the gauge since there are flat directions corresponding to the transformation $A_\mu \mapsto A_\mu^g \equiv g\, A_\mu\, g^\dagger$, where g is an element of the coset space $H \equiv \mathrm{U}(N)/\,\mathrm{U}(k)$. In order to remove the associated zero modes, we introduce the gauge fixing term and the corresponding ghost term

$$
\begin{aligned}
S_{\text{g.f.}} &= -\frac{N}{2}\,\mathrm{tr}[X_\mu, A_\mu]^2\,, && \text{(A.1)}\\
S_{\text{ghost}} &= -N\,\mathrm{tr}\left([X_\mu, \bar{c}][A_\mu, c]\right), && \text{(A.2)}
\end{aligned}
$$

where c and $\bar{c}$ are the ghost and anti-ghost fields, respectively.

We perform the integration over A_μ perturbatively by decomposing it as $A_\mu = X_\mu + \tilde{A}_\mu$, where $X_\mu \equiv X_\mu^{(n_1,n_2;k)}$. The partition function can be written as

$$
Z = \mathrm{vol}(H)\,\mathcal{N} \int \mathrm{d}\tilde{A}\,\mathrm{d}c\,\mathrm{d}\bar{c}\,\mathrm{e}^{-S_{\text{total}}}\,, \tag{A.3}
$$

where the total action $S_{\text{total}} = S + S_{\text{g.f.}} + S_{\text{ghost}}$ is given by

$$
\begin{aligned}
S_{\text{total}} &= S[X] + S_{\text{kin}} + S_{\text{int}}\,, && \text{(A.4)}\\
S_{\text{kin}} &= \frac{1}{2}\,N\ \mathrm{tr}\left(\tilde{A}_\mu[X_\lambda,[X_\lambda,\tilde{A}_\mu]]\right) + N\ \mathrm{tr}\Big(\bar{c}\,[X_\lambda,[X_\lambda,c]]\Big)\,, && \text{(A.5)}\\
S_{\text{int}} &= -N\ \mathrm{tr}\left([\tilde{A}_\mu,\tilde{A}_\nu][X_\mu,\tilde{A}_\nu]\right) - \frac{1}{4}\,N\ \mathrm{tr}\left([\tilde{A}_\mu,\tilde{A}_\nu]^2\right) + \\
&\quad + \frac{2}{3}\,i\,\alpha\,f_{\mu\nu\rho}\,N\ \mathrm{tr}\left(\tilde{A}_\mu\tilde{A}_\nu\tilde{A}_\rho\right) + N\ \mathrm{tr}\left(\bar{c}\,[X_\mu,[\tilde{A}_\mu,c]]\right). && \text{(A.6)}
\end{aligned}
$$

The volume of the coset space H, $\mathrm{vol}(H) = \mathrm{vol}(\mathrm{U}(N))/\mathrm{vol}(\mathrm{U}(k))$, can be obtained by the formula $\mathrm{vol}(\mathrm{U}(p)) = \frac{(2\pi)^{\frac{p(p+1)}{2}}}{(p-1)!\cdots 0!}$, and the normalization factor $\mathcal{N} = (2\pi N)^{-\frac{1}{2}(N^2-k^2)}$ can be obtained by following the usual gauge fixing procedure as in ref. [68]. The prefactors $\mathrm{vol}(H)$ and $\mathcal{N}$ in (A.3) are omitted in refs. [64, 65] since they are irrelevant for the discussions in those papers, but we need to keep them for the comparison with the free energy in the Yang-Mills phase discussed in section 5.

We calculate the free energy $W = -\log Z$ as a perturbative expansion $W = \sum_{j=0}^{\infty} W_j$, where $W_j = \mathrm{O}(\alpha^{4(1-j)})$. The classical part W_0 is given by eq. (2.6). Note that the kinetic term (A.5) can be written as

$$S_{\text{kin}} = N \,\mathrm{tr} \left\{ \frac{1}{2} \tilde{A}_\mu (\mathcal{P}_\lambda)^2 \tilde{A}_\mu + \bar{c}\,(\mathcal{P}_\lambda)^2 c \right\}, \tag{A.7}$$

where we have introduced the operator $\mathcal{P}_\mu$

$$\mathcal{P}_\mu M \stackrel{\text{def}}{=} [X_\mu, M]\,, \tag{A.8}$$

which acts on the space of $N \times N$ traceless matrices. The one-loop term is obtained as

$$W_1 = 2\,\mathcal{T}r' \log \left\{ N(\mathcal{P}_\mu)^2 \right\} - \log \left\{ \mathrm{vol}(H)\,\mathcal{N} \right\}, \tag{A.9}$$

where the symbol $\mathcal{T}r'$ denotes the trace in the space of $N \times N$ matrices omitting the zero modes.[1]

B. Perturbative calculations for the fuzzy $\mathrm{S}^2 \times \mathrm{S}^2$

In this section we consider the symmetric fuzzy $\mathrm{S}^2 \times \mathrm{S}^2$ taking $n_1 = n_2 = n (= \sqrt{\frac{N}{k}})$ for the solution $X_\mu = X_\mu^{(n_1,n_2;k)}$.

B.1 One-loop calculation of the free energy

In order to solve the eigenvalue problem of the operator $(\mathcal{P}_\lambda)^2$, let us introduce

$$Y_{lm}^{(a,b)} \stackrel{\text{def}}{=} \left(\hat{Y}_{l_1,m_1} \otimes \hat{Y}_{l_2,m_2} \right) \otimes \mathbf{e}^{(a,b)}\,, \tag{B.1}$$

where $\hat{Y}_{lm}$ represents n-dimensional matrix spherical harmonics, and $\mathbf{e}^{(a,b)}$ is a $k \times k$ matrix whose (a,b) element is 1 and all the other elements are zero. The indices l and m represent the double indices (l_1, l_2) and (m_1, m_2), respectively. The matrices $Y_{lm}^{(a,b)}$ form a complete basis of $N \times N$ matrices, and they have the properties

$$\mathrm{tr} \left(Y_{lm}^{(a,b)\dagger}\, Y_{l'm'}^{(a',b')} \right) = \delta_{ll'} \delta_{mm'} \delta_{aa'} \delta_{bb'}\,, \tag{B.2}$$

$$Y_{lm}^{(a,b)\dagger} = (-1)^{m_1} (-1)^{m_2} Y_{l,-m}^{(b,a)}\,. \tag{B.3}$$

Acting the operator $(\mathcal{P}_\lambda)^2$ on $Y_{lm}^{(a,b)}$, we find

$$(\mathcal{P}_\mu)^2 Y_{lm}^{(a,b)} = \alpha^2 \left\{ l_1(l_1+1) + l_2(l_2+1) \right\} Y_{lm}^{(a,b)}\,, \tag{B.4}$$

[1] Strictly speaking, we have to treat the zero modes for $k \neq 1$ more carefully. See ref. [64] for a discussion on this point. This complication, however, does not affect our conclusion concerning the large-N limit with fixed k.

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which means that $Y_{lm}^{(a,b)}$ are the eigenstates of $(\mathcal{P}_\lambda)^2$. Thus the first term in eq. (A.9) can be obtained as

$$2\,\mathcal{T}r' \log\left\{N(\mathcal{P}_\mu)^2\right\} = 2k^2 \sum_{l_1,l_2=0}^{n-1}{}' \,(2l_1+1)(2l_2+1)\log\left[N\alpha^2\left\{l_1(l_1+1)+l_2(l_2+1)\right\}\right]$$
$$\simeq N^2\left(\log\frac{16N^3}{k^2} - 3 + 4\log\bar{\alpha}\right), \tag{B.5}$$

where the large-N limit is taken in the second line with fixed k, and the symbol $\sum'$ implies that the zero modes $l_1 = l_2 = 0$ are excluded. The second term in eq. (A.9) can be obtained as

$$-\log\left\{\mathrm{vol}(H)\,\mathcal{N}\right\} \simeq N^2\left(\log N - \frac{3}{4}\right). \tag{B.6}$$

Adding the two terms, we obtain the one-loop contribution W_1 as

$$W_1 = N^2\left(\log\frac{16N^4}{k^2} - \frac{15}{4} + 4\log\bar{\alpha}\right). \tag{B.7}$$

B.2 One-loop calculation of various observables

In order to calculate various observables in perturbation theory, we need the propagators for $\tilde{A}_\mu$ and the ghosts, which are given as

$$\left\langle(\tilde{A}_\mu)_{ij}(\tilde{A}_\nu)_{kl}\right\rangle_0 = \delta_{\mu\nu}\sum_{ab}\sum_{l_1,l_2=0}^{n-1}{}'\sum_{m_1=-l_1}^{l_1}\sum_{m_2=-l_2}^{l_2}\frac{(-1)^{m_1}(-1)^{m_2}}{N\alpha^2\left\{l_1(l_1+1)+l_2(l_2+1)\right\}}\times$$
$$\times\left(Y_{lm}^{(a,b)}\right)_{ij}\left(Y_{l,-m}^{(b,a)}\right)_{kl}, \tag{B.8}$$

$$\langle(c)_{ij}(\bar{c})_{kl}\rangle_0 = \sum_{ab}\sum_{l_1,l_2=0}^{n-1}{}'\sum_{m_1=-l_1}^{l_1}\sum_{m_2=-l_2}^{l_2}\frac{(-1)^{m_1}(-1)^{m_2}}{N\alpha^2\left\{l_1(l_1+1)+l_2(l_2+1)\right\}}\times$$
$$\times\left(Y_{lm}^{(a,b)}\right)_{ij}\left(Y_{l,-m}^{(b,a)}\right)_{kl}, \tag{B.9}$$

where the symbol $\langle\,\cdot\,\rangle_0$ refers to the expectation value calculated using the kinetic term S_{kin} in (A.5) only.

We also need to obtain the tadpole $\langle\tilde{A}_\mu\rangle$, which can be expressed as

$$\langle\tilde{A}_\mu\rangle = c\,X_\mu \tag{B.10}$$

with some coefficient c due to the SO(3)×SO(3)× Z_2 symmetry. Since

$$\frac{1}{N}\,\mathrm{tr}\sum_{\mu=1}^{3}(X_\mu\langle\tilde{A}_\mu\rangle) = c\;\mathrm{tr}\sum_{\mu=1}^{3}(X_\mu X_\mu) = \frac{c}{4}\,\alpha^2\,(n^2-1), \tag{B.11}$$

we can obtain the coefficient c by evaluating the left most term in (B.11). At the leading order in $1/\alpha^4$, we obtain

$$\frac{1}{N}\,\mathrm{tr}\sum_{\mu=1}^{3}\Big(X_\mu\langle\tilde{A}_\mu\rangle_{1-\mathrm{loop}}\Big) = \left\langle \mathrm{tr}\sum_{\mu=1}^{3}\sum_{\nu,\rho=1}^{6}(X_\mu\tilde{A}_\mu)\ \mathrm{tr}\Big([\tilde{A}_\nu,\tilde{A}_\rho][X_\nu,\tilde{A}_\rho]\Big)\right\rangle_0 - \\ - \left\langle \mathrm{tr}\sum_{\mu=1}^{3}\sum_{\nu,\rho,\sigma=1}^{6}(X_\mu\tilde{A}_\mu)\ \mathrm{tr}\left(\frac{2}{3}\,i\,\alpha\, f_{\nu\rho\sigma}\tilde{A}_\nu\tilde{A}_\rho\tilde{A}_\sigma\right)\right\rangle_0 - \\ - \left\langle \mathrm{tr}\sum_{\mu=1}^{3}\sum_{\nu=1}^{6}(X_\mu\tilde{A}_\mu)\ \mathrm{tr}\Big(\bar{c}\,[X_\nu,[\tilde{A}_\nu,c]]\Big)\right\rangle_0 . \tag{B.12}$$

Using the fact that $L_\mu^{(n)}$ can be written as a linear combination of $\hat{Y}_{l=1,m}$, we can evaluate (B.12) as

$$\mathrm{tr}\sum_{\mu=1}^{3}\Big(X_\mu\langle\tilde{A}_\mu\rangle_{1-\mathrm{loop}}\Big) = -\frac{2}{N\alpha^2}k^2\sum_{l_1,l_2=0}^{n-1}{}' \frac{(2l_1+1)(2l_2+1)l_1(l_1+1)}{l_1(l_1+1)+l_2(l_2+1)} . \tag{B.13}$$

From (B.11) we obtain

$$c = -\frac{8k^2}{N^2\alpha^4(n^2-1)}\sum_{l_1,l_2=0}^{n-1}{}' \frac{(2l_1+1)(2l_2+1)l_1(l_1+1)}{l_1(l_1+1)+l_2(l_2+1)} . \tag{B.14}$$

Using the propagator and the tadpole obtained above, we can evaluate various observables at the one-loop level. For instance, the "extent of space-time" is given by

$$\left\langle \frac{1}{N}\,\mathrm{tr}(A_\mu)^2\right\rangle_{1-\mathrm{loop}} = \frac{1+2c}{2}\alpha^2(n^2-1) + \sum_{l_1,l_2=0}^{n-1}{}' \frac{6k^2(2l_1+1)(2l_2+1)}{N^2\alpha^2\left\{l_1(l_1+1)+l_2(l_2+1)\right\}} . \tag{B.15}$$

At large N with fixed $\bar{\alpha}\equiv\alpha N^{\frac{1}{4}}$, we obtain

$$\frac{1}{\sqrt{N}}\left\langle \frac{1}{N}\,\mathrm{tr}(A_\mu)^2\right\rangle_{1-\mathrm{loop}} \simeq \frac{\bar{\alpha}^2}{2k} - \frac{4}{\bar{\alpha}^2} . \tag{B.16}$$

The expectation value $\langle S\rangle$ can be calculated in a similar manner, but it is much easier to calculate it in the following way. Let us define a rescaled action

$$S(\lambda,\alpha) = \lambda\, N\ \mathrm{tr}\left(-\frac{1}{4}\,[A_\mu,A_\nu]^2 + \frac{2}{3}\,i\,\alpha\, f_{\mu\nu\rho}\,A_\mu A_\nu A_\rho\right) \tag{B.17}$$

and the corresponding free energy

$$e^{-W(\lambda,\alpha)} = \int dA\, e^{-S(\lambda,\alpha)} , \tag{B.18}$$

which is related to the original free energy $W = W(1,\alpha)$ through

$$W(\lambda,\alpha) = \frac{3}{2}\,(N^2-1)\,\log\lambda + W(1,\lambda^{1/4}\alpha) . \tag{B.19}$$

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Then we obtain the expectation value $\langle S \rangle$ as

$$\frac{\langle S \rangle}{N^2} = \frac{1}{N^2} \frac{\partial W(\lambda, \alpha)}{\partial \lambda}\bigg|_{\lambda=1} = \frac{3}{2}\left(1 - \frac{1}{N^2}\right) + \frac{1}{4N^2}\,\bar{\alpha}\,\frac{\partial W}{\partial \bar{\alpha}} \tag{B.20}$$

$$\simeq -\frac{\bar{\alpha}^4}{12k} + \frac{5}{2}\,. \tag{B.21}$$

B.3 Critical point

As we see in section 3, Monte Carlo simulations show that the fuzzy $\mathrm{S}^2 \times \mathrm{S}^2$ decays into the pure Yang-Mills vacuum at some critical point. We can reproduce the critical point by perturbative calculations as follows. Let us consider the one-loop effective action for the rescaled fuzzy $\mathrm{S}^2 \times \mathrm{S}^2$ configuration

$$A_\mu = \begin{cases} \beta\,(L_\mu^{(n)} \otimes \mathbf{1}_n) \otimes \mathbf{1}_k & \text{for} \quad \mu = 1,2,3\,, \\ \beta\,(\mathbf{1}_n \otimes L_\mu^{(n)}) \otimes \mathbf{1}_k & \text{for} \quad \mu = 4,5,6\,, \end{cases} \tag{B.22}$$

which is given at large N as

$$\frac{1}{N^2}\,\Gamma_{k\,\mathrm{S}^2\times\mathrm{S}^2}(\bar{\beta}) \simeq \left(\frac{1}{8}\,\bar{\beta}^4 - \frac{1}{6}\,\bar{\alpha}\bar{\beta}^3\right)\frac{2}{k} + 4\,\log\bar{\beta} + \log\frac{16N^4}{k^2} - \frac{15}{4}\,, \tag{B.23}$$

where $\bar{\beta} = \beta N^{\frac{1}{4}}$. The derivation is similar to the calculation of the free energy described in section B.1. (Note that one-particle reducible diagrams do not appear at the one-loop level.) In fact the effective action is one-loop exact in the large-N limit as can be shown by a power counting argument [47]. The effective action has extrema at $\bar{\beta}$ satisfying

$$\frac{1}{N^2}\,\frac{\partial \Gamma_{k\,\mathrm{S}^2\times\mathrm{S}^2}(\bar{\beta})}{\partial \bar{\beta}} = \frac{1}{k}(\bar{\beta}^3 - \bar{\alpha}\bar{\beta}^2) + \frac{4}{\bar{\beta}} = 0\,. \tag{B.24}$$

From this we find that the effective action $\Gamma_{k\,\mathrm{S}^2\times\mathrm{S}^2}$ has a local minimum if and only if $\bar{\alpha} > \bar{\alpha}_{\mathrm{cr}}^{(k\,\mathrm{S}^2\times\mathrm{S}^2)}$, where the critical point $\bar{\alpha}_{\mathrm{cr}}^{(k\,\mathrm{S}^2\times\mathrm{S}^2)}$ is given by

$$\bar{\alpha}_{\mathrm{cr}}^{(k\,\mathrm{S}^2\times\mathrm{S}^2)} = \frac{4}{3} \times (12k)^{1/4} \simeq 2.481613\ldots \times k^{1/4}\,. \tag{B.25}$$

B.4 All order results

Since the free energy and the effective action are related to each other by the Legendre transformation, we can obtain the free energy by evaluating the effective action at its extremum. Then exploiting the fact that the effective action is one-loop exact, we can evaluate the free energy to all orders in perturbation theory. This method is proposed in ref. [34], and it is applied to other models successfully [67, 65].

Above the critical point $\bar{\alpha} > \bar{\alpha}_{\mathrm{cr}}^{(k\,\mathrm{S}^2\times\mathrm{S}^2)}$, the value of $\bar{\beta}$ that gives the local minimum of the effective action can be obtained by solving eq. (B.24) as

$$\bar{\beta} = f(\bar{\alpha}) \stackrel{\mathrm{def}}{=} \frac{\bar{\alpha}}{4}\left(1 + \sqrt{1+\delta} + \sqrt{2 - \delta + \frac{2}{\sqrt{1+\delta}}}\right)\,, \tag{B.26}$$

where

$$\delta = \bar{\alpha}^{-4/3}(128k)^{1/3}\left\{\left(1+\sqrt{1-\frac{1024k}{27\bar{\alpha}^4}}\right)^{1/3}+\left(1-\sqrt{1-\frac{1024k}{27\bar{\alpha}^4}}\right)^{1/3}\right\}. \tag{B.27}$$

At large $\bar{\alpha}$, the solution (B.26) is expanded as

$$\bar{\beta} = f(\bar{\alpha}) = \bar{\alpha}\left(1-\frac{4k}{\bar{\alpha}^4}-\frac{48k^2}{\bar{\alpha}^8}-\frac{960k^3}{\bar{\alpha}^{12}}-\cdots\right). \tag{B.28}$$

Plugging this solution into (B.23), we obtain the free energy to all orders as

$$\frac{1}{N^2}W_{k\,\mathrm{S}^2\times\mathrm{S}^2} \simeq -\frac{\bar{\alpha}^4}{12k}+4\log\bar{\alpha}+\log\frac{16N^4}{k^2}-\frac{15}{4}-\frac{8k}{\bar{\alpha}^4}-\frac{224k^2}{3\bar{\alpha}^8}-\frac{3520k^3}{3\bar{\alpha}^{12}}-\cdots. \tag{B.29}$$

Using (B.20), we obtain the all order result for the expectation value $\langle S\rangle$ as

$$\frac{1}{N^2}\langle S\rangle \simeq -\frac{\bar{\alpha}^4}{12k}+\frac{5}{2}+\frac{8k}{\bar{\alpha}^4}+\frac{448k^2}{3\bar{\alpha}^8}+\frac{3520k^3}{\bar{\alpha}^{12}}+\cdots. \tag{B.30}$$

We can also calculate other observables to all orders as in ref. [65]. For instance, let us calculate $\left\langle \frac{1}{N}\,\mathrm{tr}(A_\mu)^2\right\rangle$. Since the one-loop contribution comes totally from the tadpole diagrams, which are one-particle reducible (The second term in eq. (B.15) is subleading in N.), we may obtain the all order result by simply replacing $\bar{\alpha}$ by the quantum solution $f(\bar{\alpha})$ in the classical result as

$$\frac{1}{\sqrt{N}}\left\langle \frac{1}{N}\,\mathrm{tr}(A_\mu)^2\right\rangle \simeq \frac{1}{2k}f(\bar{\alpha})^2 = \frac{\bar{\alpha}^2}{2k}-\frac{4}{\bar{\alpha}^2}-\frac{40k}{\bar{\alpha}^6}-\frac{768k^2}{\bar{\alpha}^{10}}-\frac{18304k^3}{\bar{\alpha}^{14}}-\cdots. \tag{B.31}$$

C. Perturbative calculations for the fuzzy S^2

In this section we briefly describe the perturbative calculations for the fuzzy S^2 taking $n_1 = n(=\frac{N}{k})$ and $n_2 = 1$ for the solution $X_\mu = X_\mu^{(n_1,n_2;k)}$.

C.1 One-loop calculations

The one-loop free energy is given at large N by

$$\begin{aligned}\frac{1}{N^2}W_{k\,\mathrm{S}^2} &= -\frac{\alpha^4N^2}{24}(n^2-1)+2k^2\sum_{l=1}^{n-1}(2l+1)\log[N\alpha^2 l(l+1)]-\log\Big\{\mathrm{vol}(H)\,\mathcal{N}\Big\}\\ &\simeq N^2\left(-\frac{\tilde{\alpha}^4}{24k^2}+4\log\tilde{\alpha}+\log\frac{N^5}{k^4}-\frac{11}{4}\right),\end{aligned} \tag{C.1}$$

where $\tilde{\alpha} = \alpha N^{\frac{1}{2}}$. The tadpole is given by

$$\langle\tilde{A}_\mu\rangle_{1-\mathrm{loop}} = \begin{cases} -\dfrac{8k^2}{N^2\alpha^4}X_\mu & \text{for } \mu = 1,2,3\,,\\ 0 & \text{for } \mu = 4,5,6\,.\end{cases} \tag{C.2}$$

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The observables can be evaluated as

$$\begin{aligned}
\frac{1}{N}\left\langle\frac{1}{N}\ \mathrm{tr}\ Q_1\right\rangle_{1-\mathrm{loop}} &= \frac{1}{N^2}\Big\{\mathrm{tr}(X_\mu X_\mu)+2\ \mathrm{tr}\left(X_\mu\langle\tilde{A}_\mu\rangle_{1-\mathrm{loop}}\right)+\langle\mathrm{tr}(\tilde{A}_\mu)^2\rangle_0\Big\} \\
&= \alpha^2\left\{\frac{1}{4N}(n^2-1)-\frac{4k^2(n^2-1)}{N^3\alpha^4}+\frac{6}{Nn^2\alpha^4}\sum_{l=1}^{n-1}\frac{2l+1}{l(l+1)}\right\} \\
&\simeq \frac{\tilde{\alpha}^2}{4k^2}-\frac{4}{\tilde{\alpha}^2}\,, \qquad (C.3)\\
\frac{1}{N}\left\langle\frac{1}{N}\ \mathrm{tr}\ Q_2\right\rangle_{1-\mathrm{loop}} &\simeq 0\,, \qquad (C.4)\\
\frac{1}{N^2}\langle S\rangle_{1-\mathrm{loop}} &= -\frac{\alpha^4}{24}(n^2-1)+\frac{5}{2}+\frac{1}{4N^2}\Big\{6(-k^2-1)+2k^2\Big\} \\
&\simeq -\frac{\tilde{\alpha}^4}{24k^2}+\frac{5}{2}\,. \qquad (C.5)
\end{aligned}$$

C.2 Critical point

The critical point can be obtained by considering the effective action for the rescaled fuzzy S^2 configuration

$$A_\mu=\begin{cases}\beta(L_\mu^{(n)}\otimes\mathbf{1}_k) & \text{for } \mu=1,2,3\,,\\ 0 & \text{for } \mu=4,5,6\ .\end{cases} \qquad (C.6)$$

The effective action is obtained in the large-N limit as

$$\frac{1}{N^2}\Gamma_{k\,\mathrm{S}^2}(\tilde{\beta})\simeq\left(\frac{1}{8}\,\tilde{\beta}^4-\frac{1}{6}\,\tilde{\alpha}\tilde{\beta}^3\right)\frac{1}{k^2}+4\,\log\tilde{\beta}+\log\frac{N^5}{k^4}-\frac{11}{4}\,, \qquad (C.7)$$

where $\tilde{\beta}=\beta\sqrt{N}$. The critical point is obtained as

$$\tilde{\alpha}_{\mathrm{cr}}^{(k\,\mathrm{S}^2)}=\frac{4}{3}\times 24^{\frac{1}{4}}\sqrt{k}=2.9511518\cdots\times\sqrt{k}\,. \qquad (C.8)$$

C.3 All order results

Above the critical point $\tilde{\alpha}>\tilde{\alpha}_{\mathrm{cr}}^{(k\,\mathrm{S}^2)}$, the effective action has a local minimum at

$$\tilde{\beta}=g(\tilde{\alpha})\stackrel{\mathrm{def}}{=}\frac{\tilde{\alpha}}{4}\left(1+\sqrt{1+\varepsilon}+\sqrt{2-\varepsilon+\frac{2}{\sqrt{1+\varepsilon}}}\right), \qquad (C.9)$$

$$\varepsilon=\tilde{\alpha}^{-4/3}(256k^2)^{1/3}\left\{\left(1+\sqrt{1-\frac{2048k^2}{27\tilde{\alpha}^4}}\right)^{1/3}+\left(1-\sqrt{1-\frac{2048k^2}{27\tilde{\alpha}^4}}\right)^{1/3}\right\}. \qquad (C.10)$$

The all order result for the free energy can be obtained by substituting $\tilde{\beta}=g(\tilde{\alpha})$ in (C.7).

$$\frac{1}{N^2}W_{k\,\mathrm{S}^2}\simeq-\frac{\tilde{\alpha}^4}{24k^2}+4\log\tilde{\alpha}+\log\frac{N^5}{k^4}-\frac{11}{4}-\frac{16k^2}{\tilde{\alpha}^4}-\frac{896k^4}{3\tilde{\alpha}^8}-\frac{28160k^6}{3\tilde{\alpha}^{12}}-\cdots\,. \qquad (C.11)$$

The expectation values of observables can be calculated at large N as

$$\frac{1}{N^2}\langle S\rangle \simeq -\frac{\tilde{\alpha}^4}{24k^2}+\frac{5}{2}+\frac{16k^2}{\tilde{\alpha}^4}+\frac{1792k^4}{3\tilde{\alpha}^8}+\frac{28160k^6}{\tilde{\alpha}^{12}}+\cdots, \tag{C.12}$$

$$\frac{1}{N}\left\langle\frac{1}{N}\,\mathrm{tr}\,Q_1\right\rangle \simeq \frac{\tilde{\alpha}^2}{4k^2}-\frac{4}{\tilde{\alpha}^2}-\frac{80k^2}{\tilde{\alpha}^6}-\frac{3072k^4}{\tilde{\alpha}^{10}}-\frac{146432k^6}{\tilde{\alpha}^{14}}-\cdots, \tag{C.13}$$

$$\frac{1}{N}\left\langle\frac{1}{N}\,\mathrm{tr}\,Q_2\right\rangle \simeq 0\,. \tag{C.14}$$

D. Free energy in the Yang-Mills phase

In ref. [71] the free energy for the pure Yang-Mills model ($\alpha = 0$) is calculated as

$$\frac{1}{N^2}F \simeq D\left\{f_D+\frac{1}{2}\ln\left(\sqrt{\frac{D}{2}}N\right)-\frac{1}{2}\ln\pi\right\}, \tag{D.1}$$

where D is the number of bosonic matrices, which is $D = 6$ in the present case, and the third term is added to adjust the normalization of the measure to the one used in the present paper. The "free energy density" f_D can be calculated analytically by the $1/D$ expansion [49], and one obtains $f_\infty = -1/4$ at $D = \infty$. For $D = 6$ the gaussian expansion method [71] gives $f_6 = -0.33$ (the convergence is clear up to order 7). Therefore the free energy in the Yang-Mills phase of our model is given at large N as

$$\frac{1}{N^2}W_{\mathrm{YM}} \simeq 3\log N + 6\left\{-0.33+\log\left(3^{1/4}\pi^{-1/2}\right)\right\}, \tag{D.2}$$

if α is sent to zero in the large-N limit. Finite α corrections should be $\mathrm{O}(\alpha^2)$ due to the parity symmetry of the pure Yang-Mills model.

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